Noncommutative $D = 5$ Chern-Simons gravity: Kaluza-Klein Reduction and Chiral Gravitation Anomaly

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Context

Introduction

In a four dimensional gauge field theory, based on a gauge symmetry principle, one can add to the Lagrangian a term $\mathrm{Tr}(F \wedge F)$. As it is a total derivative, it has no influence on a classical equations of motion. But it opens another possibility. As

$$\mathrm{Tr}(F^2) = \mathrm{d}(...),$$

we ca use $(...)$ as a Lagrangian of a three dimensional gauge theory. This construction gives rise to the Chern-Simons (CS) gauge theory, that has been thoroughly considered by physical and physical mathematics community in the past half of a century. Here we will be interested only in a limited set of properties of CS theory, and we will not review those of a minor relevance to our work.

Let us be slightly more formal. Let A be a (local) connection, defined as a pullback of a Ehresmann connection by a local section ($A = s^*\omega$). We will assume that our principal bundle is trivial, and therefore A can be considered as a (globally) Lie algebra valued one form. Let F be a curvature two form $F = \mathrm{d}A + A^2$ (most of the time we suppress wedge product). Chern-Weill theorem asserts that (Tr can be a symmetric invariant tensor of the algera, but we will use a trace in an explicit matrix representation)

$$\mathrm{Tr}(F^n) = \mathrm{d}Q_{CS}^{(2n-1)}, \tag{1}$$

where we have

$$Q_{CS}^{(2n-1)} = n\int_0^1 t^{n-1}\mathrm{Tr}\left(A(F+(t-1)A^2)^{n-1}\right)\mathrm{d}t. \tag{2}$$

Introduction

We define an action

$$S_{CS}^{(2n-1)} = \alpha \int\limits_{\mathcal{M}_{2n-1}} Q_{CS}^{(2n-1)} = \int\limits_{\mathcal{M}_{2n-1}} L_{CS}^{(2n-1)}. \tag{3}$$

We will work with $n = 3$. In this case, we have

$$S_{CS}^{(5)} = \alpha \int_{\mathcal{M}_5} F^2 A - \frac{1}{2} F A^3 + \frac{1}{5} A^5.$$

In four dimensions, the most general action in general relativity is given by an Einstein-Hilbert action with a cosmological constant (*Lovelock's theorem*). We will not work use the settings of GR. We will work in the first order formalism (vielbein and spin connection are treated independently), and we will start from five dimensions. Generalisation of Lovelock theorem allows us to consider the most general action (without explicit torsion)

$$L_{LL}^{(D)} = \sum_{i=0}^{[D/2]} c_i L_i^{(D)}, \tag{4}$$

where $L_i^{(D)} = \varepsilon_{a_1 \dots a_D} R^{a_1 a_2} \dots R^{a_{2i-1} a_{2i}} e^{a_{2i+1}} \dots e^{a_D}$.

Gravitation

In five dimensions, there is a term $\sim \varepsilon_{abcde} R^{ab} R^{cd} e^{e}$ (Gauss-Bonnet, not topological here). For suitable choice of c_i, local symmetry is enchanced, and we have local AdS/dS/Poincare symmetry. This can be seen by rewriting the action as a CS action for a suitable Lie algebra. We will work in the AdS case. Lie algebra $\mathfrak{so}(4,2)$ has generators:

$$[J_{\mathbb{AB}}, J_{\mathbb{CD}}] = G_{\mathbb{AD}} J_{\mathbb{BC}} + G_{\mathbb{BC}} J_{\mathbb{AD}} - (\mathbb{C} \leftrightarrow \mathbb{D}), \tag{5}$$

where $G_{\mathbb{AB}} = (-++++-)$. By splitting indices, we have

$$\begin{aligned} [J_{AB}, J_{CD}] &= G_{AD} J_{BC} + G_{BC} J_{AD} - (C \leftrightarrow D), \\ [J_{AB}, J_{C5}] &= G_{BC} J_{A5} - G_{AC} J_{B5}, \\ [J_{A5}, J_{C5}] &= J_{AC}, \end{aligned} \tag{6}$$

$\mathfrak{so}(4,2)$ algebra representation

Though Tr can be more general, we will use a concrete representation of $\mathfrak{so}(4,2)$ algebra, given by

$$J_{AB} = \frac{1}{4}[\Gamma_A, \Gamma_B] \qquad A = 0,1,2,3,4$$

$$J_{A5} = \frac{1}{2}\Gamma_A$$

$$\Gamma_a = -i\gamma_a \qquad a = 0,1,2,3$$

$$\Gamma_4 = \gamma_5,$$

Gamma matrices in lower letters correspond to a "reversed" signature, four dimensional gamma matrices, usually used in QFT.

Using mentioned representation, we compute $\mathrm{Tr}\left(F^3\right) = \frac{3i}{8}\varepsilon_{ABCDE}F^{AB}F^{CD}F^{E5}$. Taking $A = \frac{1}{2}\Omega^{AB}J_{AB} + \frac{1}{l}E^{A}J_{A5}$ (dimensional analysis implies that parameter l has to be introduced),we get

$$S^{(5)}_{CS} = \frac{k}{8}\int \varepsilon_{ABCDE}\left(\frac{1}{l}R^{AB}R^{CD}E^{E} + \frac{2}{3l^3}R^{AB}E^{C}E^{D}E^{E} + \frac{1}{5l^5}E^{A}E^{B}E^{C}E^{D}E^{E}\right). \tag{7}$$

Constant k is introduced as $\alpha = -ik/3$. We can rewrite this as

$$S^{(5)}_{CS} = \frac{1}{16\pi G^{(5)}}\int \mathrm{d}^5x\sqrt{-g}\Big[R - 2\Lambda + \frac{l^2}{4}\left(R^2 - 4R^{\mu\nu}R_{\nu\mu} + R^{\mu\nu\rho\sigma}R_{\rho\sigma\mu\nu}\right)\Big] \tag{8}$$

Short review of NC field theory

We will be interested in:

- Canonical noncommutativity $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where $\theta^{\mu\nu} = const.$
- We deform an algebra of functions

$$f \star g = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x^\mu}\partial_{y^\nu}} f(x)g(y)|_{x\to y}$$

- It is more natural to deal with CS Lagrangians using differential forms; we introduce an Abelian Drinfeld Twist

$$\tau_p \wedge_\star \tau'_q \equiv \sum_{n=0}^{+\infty} \left(\frac{i}{2}\right)^n \theta^{I_1 J_1} \dots \theta^{I_n J_n} (\ell_{I_1} \dots \ell_{I_n} \tau_p) \wedge (\ell_{J_1} \dots \ell_{J_n} \tau_q)$$

  Lie derivatives are taken with respect to a set of mutually commuting vector fields X_I.

In a commutative gauge theory, we have

$$\delta_\epsilon A = -\mathrm{d}\epsilon - [A, \epsilon] = -\mathrm{d}\epsilon - A \wedge \epsilon + \epsilon \wedge A, \tag{9}$$

$$\delta_\epsilon F = [\epsilon, F] = \epsilon \wedge F - F \wedge \epsilon, \tag{10}$$

where $\epsilon = \epsilon^K T_K$ is a gauge transformation parameter. For NC fields, we define

$$\widehat{\delta}_{\widehat{\epsilon}} \widehat{A} = -\mathrm{d}\widehat{\epsilon} - \widehat{A} \wedge_\star \widehat{\epsilon} + \widehat{\epsilon} \wedge_\star \widehat{A}, \tag{11}$$

$$\widehat{\delta}_{\widehat{\epsilon}} \widehat{F} = \widehat{\epsilon} \wedge_\star \widehat{F} - \widehat{F} \wedge_\star \widehat{\epsilon}. \tag{12}$$

We note that

$$[\widehat{\delta}_{\widehat{\epsilon}_1}, \widehat{\delta}_{\widehat{\epsilon}_2}] \widehat{F} = \widehat{\delta}_{-[\widehat{\epsilon}_1, \widehat{\epsilon}_2]_\star} \widehat{F} = -[[\widehat{\epsilon}_1, \widehat{\epsilon}_2]_\star, \widehat{F}]_\star, \tag{13}$$

where

$$[\widehat{\epsilon}_1, \widehat{\epsilon}_2]_\star = \frac{1}{2} \left([\widehat{\epsilon}_1^K, \widehat{\epsilon}_2^L]_\star \{T_K, T_L\} + \{\widehat{\epsilon}_1^K, \widehat{\epsilon}_2^L\}_\star [T_K, T_L] \right). \tag{14}$$

Anticomutator in the last expression implies that the commutator does not close in the algebra. In order to cure this problem, one introduces an Enveloping algebra. Unfortunately, this introduces (infinity) new degrees of freedom. Seiberg and Wittern showed that it is possible to express all the degrees of freedom in terms of the commutative ones, in a perturbative expansion in θ. Thus, NC gauge field theory is connected with a commutative gauge field theory. We demand:

$$\widehat{\delta}_{\widehat{\epsilon}}\widehat{A}(A) = \widehat{A}(A+\delta_{\epsilon}A) - \widehat{A}(A), \tag{15}$$

Then one can obtain (keeping only linear terms in θ):

$$\hat{A} = A - \frac{i}{4}\theta^{IJ}\{A_I, \ell_J A + F_J\}, \tag{16}$$

$$\hat{\epsilon} = \epsilon - \frac{i}{4}\theta^{IJ}\{A_I, \ell_J \epsilon\}. \tag{17}$$

NC Correction

In the last formula, A_I stands for contracted connection along the direction of X_I. NC CS action in $5D$ reads

$$S^{(5)}_{CS,NC} = -\frac{ik}{3}\int \mathrm{Tr}\Big(\widehat{F}\wedge_\star \widehat{F}\wedge_\star \widehat{A} \\ -\frac{1}{2}\widehat{F}\wedge_\star \widehat{A}\wedge_\star \widehat{A}\wedge_\star \widehat{A} \\ +\frac{1}{10}\widehat{A}\wedge_\star \widehat{A}\wedge_\star \widehat{A}\wedge_\star \widehat{A}\wedge_\star \widehat{A}\Big). \qquad (18)$$

θ independent part is precisely the action we already considered. We again use $\mathfrak{so}(4,2)$ algebra and decompose (classical) connection as before, to obtain the first order correction of this action.

NC Correction

The first order correction of our action is given by [Aschieri, Castellani, '14]

$$S^{(5)}_{CS,\theta} = \frac{k\theta^{IJ}}{12} \times \int \Big(F^{AB}(F_I)_{BC}(\mathrm{D}_\Omega F_J)^C{}_A + \frac{1}{l^2} F^{AB}(F_I)_{BC}(T_J)^C E_A$$
$$+ \frac{1}{l^2} F^{AB}(T_I)_B(\mathrm{D}_\Omega T_J)_A + \frac{2}{l^2} F^{AB}(T_I)_B(F_J)_{AC} E^C$$
$$+ \frac{1}{l^2} T^A (T_I)^B (\mathrm{D}_\Omega F_J)_{BA} + \frac{1}{l^2} T^A (\mathrm{D}_\Omega T_I)^B (F_J)_{BA}$$
$$+ \frac{1}{l^2} T_A (F_I)^{AB}(F_J)_{BC} E^C + \frac{2}{l^4} T_A (T_I)_B (T_J)^{[B} E^{A]} \Big).$$

KK Reduction

Chamseddine showed that CS gravity can be connected to a topological $\sim \mathrm{Tr}\phi F^n$ gravity by a dimensional reduction (with truncation).

$$
\begin{aligned}
S_{red} = -\frac{k(2\pi R)}{8} \int \mathrm{d}^4x \; \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd} \\
\Bigg(\frac{1}{l^5} E^a_\mu E^b_\nu E^c_\rho E^d_\sigma E^4_4 - \frac{4}{l^5} E^a_\mu E^b_\nu E^c_\rho E^4_\sigma E^d_4 + \frac{2}{l^3} E^a_\mu E^b_\nu E^c_4 R^{d4}_{\rho\sigma} \\
+ \frac{1}{l^3} E^a_\mu E^b_\nu E^4_4 R^{cd}_{\rho\sigma} - \frac{2}{l^3} E^a_\mu E^4_\nu E^b_4 R^{cd}_{\rho\sigma} + \frac{4}{3l^3} E^a_\mu E^b_\nu E^c_\rho R^{d4}_{\sigma 4} \\
+ \frac{2}{l^3} E^a_\mu E^b_\nu E^4_\rho R^{cd}_{\sigma 4} + \frac{1}{l} E^a_4 R^{bc}_{\mu\nu} R^{d4}_{\rho\sigma} + \frac{1}{4l} E^4_4 R^{ab}_{\mu\nu} R^{cd}_{\rho\sigma} \\
+ \frac{2}{l} E^a_\mu R^{bc}_{\nu\rho} R^{d4}_{\sigma 4} + \frac{2}{l} E^a_\mu R^{b4}_{\nu\rho} R^{cd}_{\sigma 4} + \frac{1}{l} E^4_\mu R^{ab}_{\nu\rho} R^{cd}_{\sigma 4} \Bigg).
\end{aligned}
\tag{19}
$$

KK reduction

We truncate components Ω_4^{ab}, E_4^a, Ω_μ^{a4}, и E_μ^4 (consistent with the residual $SO(3,2)$ symmetry). We furthermore label $\Omega_\mu^{ab} \equiv \omega_\mu^{ab}$, $E_\mu^a \equiv e_\mu^a$ and $\phi^a \equiv -l^2 \Omega_4^{a4}$, $\varphi \equiv l E_4^4$. We can arrange remaining fields as

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2}\omega^{ab} J_{ab} + l^{-1} e^a J_{a5}, \\
\mathcal{F} &= \frac{1}{2} F^{ab} J_{ab} + F^{a5} J_{a5}, \\
\Phi &= \Phi^a J_{4a} + \Phi^5 J_{45} = \phi^a J_{4a} + \varphi J_{45},
\end{aligned} \tag{20}$$

in order to write

$$S_{red} = \frac{ik(2\pi R)}{l^2} \int \mathrm{Tr}\left(\mathcal{F}\mathcal{F}\Phi\right) \tag{21}$$

Action (21) posses a manifest $SO(3,2)$ symmetry.

Obtained theory of gravity can be considered, but due to it's enlarged local symmetry group, one cannot immediately connect it with observable physics. We therefore insist on a symmetry breaking of $SO(3,2)$ down to $SO(3,1)$, that is done (motivated by MacDowell-Mansouri-Chamseddine-Stelle-West approach) by $\phi^a = 0$ и $\varphi = l$. We are left with

$$S_{red} = \frac{k(2\pi R)}{8l^3} \int \varepsilon_{abcd} \times \left(l^2 R^{ab} R^{cd} + 2R^{ab} e^c e^d + \frac{1}{l^2} e^a e^b e^c e^d \right). \tag{22}$$

This is Einstein-Hilbert action with cosmological constant (together with the topological Gauss-Bonnet term), written in the first-order formalism.

Noncommutative version

We apply the mentioned procedure to the action obtained as a first order correction to a classical CS gravity action. After calculation, the final result is

$$
\begin{aligned}
S_{red,NC} &= S_{red} + \frac{(2\pi R)k}{12}\theta^{I4} \\
&\times \int \left[\frac{2}{l^4} R^{ab} T_a (e_I)_b - \frac{4}{l^4} T^a (R_I)_{ab} e^b \right. \\
&\left. + \frac{2}{l^4} R^{ab} (T_I)_a e_b + \frac{6}{l^6} T^a e_a (e_I)^b e_b \right] \qquad (23)
\end{aligned}
$$

We assumed $\partial_\mu X_I^4 = 0$ (consistent from the point of view of a $4D$ theory). We defined $\theta^{I4} \equiv \theta^{IJ} X_J^4$.

Equations of motion

Varying action with respect to e^a and ω^{ab}, we obtain

$$
\begin{aligned}
\delta e_d: \quad & \varepsilon_{abc}{}^{d}\left(R^{ab}e^c + \frac{1}{l^2}e^a e^b e^c\right) \\
& -\frac{\theta^{I4}}{3l}\left[\left(R^{db} + \frac{3}{l^2}e^d e^b\right)(\mathrm{D}_\omega e_I)_b - 2(\mathrm{D}_\omega R_I)^{db}e_b\right] = 0, \qquad (24) \\
\delta\omega_{ac}: \quad & \varepsilon^{ac}{}_{bd}T^b e^d + \frac{\theta^{I4}}{3l}\left[\frac{1}{2}R^{ab}e^c(e_I)_b\right. \\
& \left. -\frac{1}{2}R^{cb}e^a(e_I)_b + (R_I)^{ab}e^c e_b - (R_I)^{cb}e^a e_b + \frac{3}{l^2}e^a e^b e^c (e_I)_b\right] = 0. \qquad (25)
\end{aligned}
$$

As θ is a small parameter, and we used only first order correction of an action, we solve equations (24) and (25) perturbatively.

AdS Spacetime

Line element in a suitable coordinates is given by

$$\mathrm{d}s^2 = -\left(1+\frac{r^2}{l^2}\right)\mathrm{d}t^2 + \frac{1}{\left(1+\frac{r^2}{l^2}\right)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \qquad (26)$$

Classically, torsion is zero. We make an ansatz $e^a \to e^a + \tilde{e}^a$ and $\omega^{ab} \to \omega^{ab} + \tilde{\omega}^{ab}$. It is not hard to see that the corrections are actuall zero in this case, and AdS spacetime remains a solution of E.O.M even after introducing the first order correction.

AdS Schwarzschild Black Hole

Line element is given by

$$ds^2 = -f^2(r)dt^2 + \frac{1}{f^2(r)}dr^2 + r^2 d\Omega^2, \qquad (27)$$

where $f^2(r) \equiv \left(1 - \frac{2m}{r} + \frac{r^2}{l^2}\right)$. It turns out that it is enough to assume that only spin connection receives a correction. In order to simplify our equations, we will assume that only two vector fields X_I are given by ∂_r and ∂_4. First, we obtain torsion components

$$\begin{aligned} \tilde{T}^0_{23} &= -\frac{m\theta^{14}}{l}\frac{\sin\theta}{rf(r)}, \\ \tilde{T}^2_{03} &= \frac{m\theta^{14}}{2l}\frac{\sin\theta}{r^2}, \\ \tilde{T}^3_{02} &= -\frac{m\theta^{14}}{2lr^2}. \end{aligned} \qquad (28)$$

Next, we compute the components of a spin connection and curvature

$$\tilde{\omega}_0^{23} = \frac{m\theta^{14}}{lr^3}, \qquad \tilde{\omega}_2^{03} = -\frac{m\theta^{14}}{2lr^2 f(r)},$$
$$\tilde{\omega}_3^{02} = \frac{m\theta^{14}}{2lr^2 f(r)} \sin\theta. \tag{29}$$

$$\begin{aligned}
\tilde{R}_{23}^{01} &= -\frac{m\theta^{14}}{l}\frac{\sin\theta}{r^2},\\
\tilde{R}_{13}^{02} &= -\frac{m\theta^{14}}{l}\left[\frac{f'(r)}{2r^2} + \frac{f(r)}{r^3}\right]\frac{\sin\theta}{f^2(r)} = -\sin\theta\,\tilde{R}_{12}^{03},\\
\tilde{R}_{03}^{12} &= \frac{m\theta^{14}}{l}\left[\frac{f'(r)}{2r^2} - \frac{f(r)}{r^3}\right]\sin\theta = -\sin\theta\,\tilde{R}_{02}^{13},\\
\tilde{R}_{01}^{23} &= \frac{3m\theta^{14}}{lr^4}.
\end{aligned} \tag{30}$$

AdS Schwarzschild Black Hole

So, even though metric ($g_{\mu\nu} = e^a_\mu e_{a\nu}$) remains the same as in the commutative limit, this solution develops a torsion. We can ask what are the differences between this solution, and the classical one. In order to do this, we compute forms connected with the topological invariants. Euler form ($\varepsilon^{abcd} R_{ab} R_{cd}$) and Nieh-Yan ($T^a T^b - R^{ab} e_a e_b$) form remain the same as in the commutative case. On the other hand, Pontryagin form, that is classically zero, is now given by

$$R^{ab} R_{ab} = \frac{48 m^2 \theta^{14}}{l r^5} \sin\theta \, \mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi. \quad (31)$$

We work for $r > r_h$, but this expression is finite at the horizon.

Chiral Gravitational Anomaly

Last result is significat, because it implies the presence of a chiral gravitational anomaly (gravity analogy of a $\psi \to e^{i\alpha\gamma^5}\psi$ anomaly in a massless electrodynamics). Upon quantisation of a massless Dirac fermion on a curved background, axial current, that is classicaly conserved, satisfies

$$\mathrm{d} * j_5 = \frac{1}{96\pi^2} R^{ab} R_{ab}. \tag{32}$$

We have

$$\mathrm{d} * j_5 = \frac{m^2\theta^{14}}{2\pi^2 l r^5} \sin\theta \, \mathrm{d}t \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{33}$$

or written in another form:

$$\partial_\mu(\sqrt{-g} j_5^\mu) = \frac{m^2\theta^{14}}{2\pi^2 l r^5} \sin\theta. \tag{34}$$

Conclusion

- $5D$ CS action has a first order in θ correction.
- This theory can be related with the four dimensional gravity by KK reduction and SB.
- Remaining action is fairly simple, and E.O.M. are solved by AdS spacetime.
- AdS Schwarzschild spacetime developes a torsion, and introduces a chiral gravitational anomaly if one couples a massless Dirac fermion to this fixed NC background.

End

Thank you for your attention!